ROLE OF SYMMETRY IN OPTIMIZATION OF FEM SIMULATION CALCULATIONS

TOMAS PERNA

MATERIAL AND METALLURGICAL RESEARCH, Ltd., Vitkovice, Ostrava, Czech Republic

DOI: 10.17973/MMSJ.2022_06_2022059

projekty@mmvyzkum.cz

At the studying the Mannesmann piercing process, we unify two approaches to the problem solving. Namely, the commercial FEM software procedure and the mathematical model of the process via the mathematical model of the considered FEM-simulation bound by certain unifying symmetries. Such phenomenon seemingly exists only if the FE-mesh is initiated to be physically interpreted. We shortly outline, how to come to the slightly modified Cahn-Hilliard equation as to the mathematical model of the FE-simulation possessing quasi-symmetry given by a lattice of colloidal assembly formed by the chosen FE-mesh. Separation of two cylindrical surfaces of the pierced product together with the inpainting role of the piercing plug are described with respect to the background given by the Navier-Stokes equations related to the flow between the both surfaces. Influence of the involved groups related to the considered quasi-symmetry is illustrated by the convergence/divergence of the Newton-Raphson number in the CPU-time.

KEYWORDS

Finite element method (FEM), numerical simulation, algorithm, mathematical modelling, Mannesmann piercing process, group theory, icosahedral group, Cahn-Hilliard equation, Navier-Stokes equation, colloidal crystal

1 INTRODUCTION

The Mannesmann process (MMP) is used at the tube production and, in a certain sense, it can be understood as a transformation of the full cylinder into the hollow one taking part between two rotating working rolls by means of the piercing plug at a certain forming temperature. The critical point is a cavity origin (the Mannesmann effect), which is an atomic-level phenomenon. In that sense it is described as a process controlled by the Schrödinger equation that is further controlled by its generalization, namely by the Klein-Gordon one for the tunnelling the potential barrier [Perna 2020]. To simulate the MMP using the FEM simulation is therefore hardly limited by the possibilities of simulation mesh selection, which cannot reach up to the atomic-level of cavity formation instances (Mannesmann effect is moreover still not completely understood, even in general [Počta 1976]). So, any FEM-simulation of the MMP cannot be realistic as the whole, but only quasi-realistic, as we will show further (the relevant results from a commercial software FORGE (TRNASVALOR S.A.) being used generally for the forging and rolling simulations are considered here) - It is namely possible to bypass this fundamental shortcoming [Berazategui 2006] by creating a preliminary hole in the product on which the MMP is intended to be simulated. In order to find out, how such a hole and the piercing plug should be combined, the mathematical model of the FEM-simulation itself must be developed. The solutions of such model should enable a piercing plug a certain involving within the simulation process as if it would be coupled with the sequence of instances of cavity formation. That´s why we will consider such a model only as the problem of a quasi-plug-action and call it as the "masked plug problem" symbolically. The region "damaged" by the preliminary hole will be again filled (reconstructed with respect to the plug) using information from surrounding areas. These areas are formed by the FE-mesh however, so that there is no physical, crystallographicaly based information to be processed. We will yield the required information from the FE-mesh interpreted in terms of the colloidal crystal assembly, admitted by the singular solution of the Navier-Stokes equation for the colloidal solution flow between two rotating cylinders that are formed by the process of the phase separation described by the Cahn-Hilliard equation. In other words, creating a mathematical model of the FEM-simulation of the MMP, we will build the Cahn-Hilliard-Navier-Stokes-system enabling the information transfer necessary for the considered reconstruction. Additively, specific symmetries are emerged with only one unique quasi-symmetry of the colloidal crystal considered. Surprisingly enough, this approach is bounded by its algorithmically characteristic polynomial of the fifth degree, so (via an existence of icosahedral iterative scheme of the solution of quintic) it indicates an existence of the relevant iteration schemes diverging with respect to the simple alternating group A_5 and its corresponding direct product with a cyclic group Z_2, i.e. $A_5 \times Z_2$, the icosahedral symmetry groups. Their *order* 60 and 120 respectively is demonstrated by the divergence of the process related Newton-Raphson number in the CPU-time. – Consequently, for the convergence case, the icosahedral broken symmetry is substituted by the unique quasi-symmetry represented by a star of vector from the lattice of the colloidal crystal coupling the correct FE-mesh choice with an admissible calibration of the MMP. Furthermore, this quasi-symmetry connects the mathematical model of the MMP-FEM-simulation with its real, atomic-level-built model by the systematic exclusion of any A_5 engagement from the correct realistic course of the MMP.

2 MATHEMATICAL MODEL ALGORITHM OF THE FEM-SIMULATION OF MMP

Preferably use DIN A4 page format and MS Word editor, please. This algorithm is represented by such an arrangement of partial differential equations (PDE) that can "substitute" a mathematical model of an atomic level Mannesmann effect within a FEM-simulation course emerging certain symmetries controlling the whole scheme consequently. Classical eigenvalue problem for the biharmonic operator (see [Pereira 2008])

$$(\Delta^2 + \lambda)u = 0 \; in \; \Omega_a \text{ bounded in } R^n \qquad (1)$$

$$u = \partial u/\partial N = 0 \text{ on } \partial\Omega_a \text{ for a certain } a \in (0, ½),$$

labelled with the „least rational" *a* which cannot be contained in any pair of dyadic (binary) rationals generating some locally cyclic group [Tsankov 2011]. – If *H* is a certain cyclic group, so it is also locally cyclic, then the rational number a representing the hole/plug diameters ratio, should not prevent the region Ω_a to be purely *H* – symmetric by some own „symmetry contribution" of *a*. Since there is a „smallest" cyclic group Z_2, which, in no way, can be generated by any pair of binary rationals, we put for *H* (see [Pereira 2008]) that

$H = Z_2 \oplus \ldots Z_2$.

For $U = gu + \ldots gu$, $g \in H$. (2)

Now, incorporating *U* within mathematical models in [Perna 2020] with respect to *H*, we consider more generally a concept of binarity, if it can be assumed that λ also satisfies the Cahn-Hilliard equation [Burger 2009] (modelling generally the phase separation in binary compositions) as

$$\partial_t U = -\Delta(\nabla\bullet(\nabla U/|\nabla U|)) + \lambda(u-U) \text{ in } \partial\Omega_a \times (0,T), \quad t \in (0,T) \subset R. \tag{3}$$

Further, using an operator ∂_j from the affine connection ∇_{∂_j} instead of the divergence operator $\nabla\bullet$ in (3), we obtain

$$\partial_t U = -k_j \Delta\left(\partial_j\left(\frac{\nabla U}{|\nabla U|}\right)\right) \text{ in } R^3 \times R \tag{4}$$

with a „hidden incorporation „of λ through an extension of the relation (3) as it should look like in $R^3 \times R$. In this way, we achieve a state with the *„masked plug*" incorporated in the process of FEM-simulation of the MMP. The last form of the Cahn-Hilliard equation takes the internal and external surfaces of the „computationally pierced product" as a binary composition with respect to additionally emerged quantity k_j that could be interpreted as a characteristic vector from a colloidal lattice, analogously like in [Tateno 2019]. It serves here as a symbol of the configuration of the perturbation mode leading to cavity formation that starts the phase separation process. This configuration (the „star of the vector k_j"[Litzman 1982]) can be found via the singular set of the Navier-Stokes equations solved for a colloidal solution Taylor-Couette flow [Larignon 2009] by means of the system TP-Complex [Perna 1995]. The network of the colloidal crystal grain boundaries is obtained as the simplest reinterpretation of the FE-mesh in the physical reality.

3 RESULTS

We call the solution *U* of (4) as the fiber of union of disjoint stars $S_\alpha(k_j)$ of the vector k_j for $\alpha = 1, 2, \ldots, M$, where *M* is a number of instances of perturbation modes representing moments of cavity origins. (Despite a certain apparent similarity with central series of the so-called nilpotent groups, it must not be confused with it.)

$$\partial(R^n \times S^1) := S^n \tag{5}$$

as the *n*-sphere (with a measure *z*) equipped with a nonconvex cover U. This cover represents a space, where the pierced product is formed at a certain value of $n \neq N$ given in our algorithm by the skew polynomial

$$P_N(z) = cos(N\, asin(z)). \tag{6}$$

Then, for a function $V = P_N(z)$, we solve the differential equation

$$\partial_j\, \partial_j\, V = 0 \;\; in\; R^3 \times R \tag{7}$$

where V takes a form of a vortex on U, finding

$$N = atan\left[\left(z\,(s-1)(1-z^2)^{\frac{3}{2}}\right)/\,(z^4(2b-s+1) + z^2(3(s-1)-4b) + 2(b-s+1))\right]/asin(z) \tag{8}$$

as a degree of correspondingly relevant groups, when Im(z)≠0 and Re(z) = ½.

So, besides a problem-calibration coupling parameter *b*, we also obtain a torsion of the group H with s ∈ Tor H and can put

$$Ln\, U = z^4(2b-s+1) + z^2(3(s-1)-4b) + 2(b-s+1) \tag{9}$$

for the Ansatz in the problem (4). Together with the „masked plug", the group H be-comes trivial by an introducing

$$f(v(s)) = f_a(v)\,, f_a : H \to 0 \tag{10}$$

there. Thus, having an internal and external surfaces of the „computationally pierced" semi-product separated (as a picture of the phase separation without torsion). In this way, s is not in Tor H,if the trivial group 0 starts to act in (10).Roughly speaking, having no crystallography within the binary separated phases assembly, we need something that is "systematically physically trivial". Here we assume that the real meaning of the trivial group 0 can be found, if the equation (4) is considered as being solved with respect to a shear viscosity ν in a "trivially crystallographic" colloidal crystal assembly (as a solid), which is quasi-provided with „finite elements-grain boundaries ".. We yield namely

$$U = exp\,[(3\nu-4)/(\chi(\nu-2)) - (3\nu-4)/(\chi^2(\nu-2)) - 2/(3(3\nu-4)) - 2/3] \tag{11}$$

with

$$f_a = exp\,[\nu/(\chi(\nu-2)) - 4/(3(3\nu-4)) - 1/3]\,, \tag{12}$$

where χ is the so-called Euler characteristic [Meyer 2005] and the constant 1/3 is emerged as the value of *a* used for the given FEM-simulation of the MM-process. The extension (4) of the problem formulation (3) can, under a certain value of the coupling parameter (constant *b*) and the FE-mesh choice, start the divergence in the simulation process itself, when $\chi \to 2$ if $t \to \infty$. Consequently, the corresponding substitutions in (11) and (12) imply

$$U \to Aut(G) \Leftrightarrow exp\,[(5\nu-4)/(6(3\nu-4)(\nu-2)) + 5/60] \tag{13}$$

$$f_a \to Inn(G) \Leftrightarrow exp\,[(5\nu-4)/(3(3\nu-4)(\nu-2)) + 5/30] \tag{14}$$

with the quotient

$$Aut(G)/Inn(G) := Out(G) = (Aut(G))^{-1}, \tag{15}$$

which is the outer automorphism group of the group *G*. The ratio 5/60 induces that $G = A_5$, the finite simple group of degree 5 and order 60. As we know, this group has only two normal subgroups, the trivial group and itself. So, the directly connected logical meaning of the trivial group 0 in (10) can be more specially seen in an existence of the trivial center Z(A_5), or in the fact that the group A_5 is centerless respectively. Then, in this extreme interpretation, *Out*(*G*) results in a form of $(Aut(G))^{-1}$ as being dual to Z(A_5) at the case of divergence of the FEM-simulation process. It is one of the exception from the rule [Wikipedia 2021].

$$N \geq 3, N \neq 6 : Out(A_N) = C_2 \text{ (see also the case: } z = Re(z) = ½\; in\; (8)). \tag{16}$$

Here C_2 is the cyclic group of the order 2. So, we consequently proceed to an existence of the cyclic group Z_2 in the direct product

$$I_h = A_5 \times Z_2\,, \tag{17}$$

where I_h is the full icosahedral group of the order 120. It is a group of symmetry of the regular icosahedron or dodecahedron respectively.

Now, retrospectively, which role does icosahedral symmetry play within the solution (U, f_a) of the problem (4)? – An answer

lies in the fact that, solving this problem, we must respect an emergency of existence condition

$$T = F_{\phi}(f_a, P(\chi)), \tag{18}$$

with ϕ as the perturbation mode coupled with the cavity formation and the polynomial $\boldsymbol{P}(\chi)$. We assume with respect to (17) that, since this polynomial is *quintic* having a form

$$\begin{aligned} P(\chi) = {} & 3\chi^5(\nu - 2)^2(5\nu - 8) - 12\chi^4(\nu - 2)^2 \\ & + \chi^3(2 - \nu)(223\nu^2 - 672\nu + 496) \\ & + 2\chi^2(\nu - 2)(3\nu - 4)(21\nu - 20) \\ & + 4\chi(3\nu - 4)^2(9\nu - 16) - 8(3\nu - 4)^3 \end{aligned} \tag{19}$$

it is not solvable by means of any icosahedral iterative scheme [Doyle 1989]. – It holds namely that according to [Doyle 1989] the generally convergent purely iterative algorithm assigns to its input data *v* a rational map $T_v(z)$, such that $T_v^n(z)$ converges for almost all *v* and *z* (the limit point is the output of the algorithm). In our case *v* are χ for uncertain ν which does not satisfy the rigidity condition necessary for the convergence of the algorithm. This "lack of rigidity" can be then nicely observed in the realistically simulated picture of a metal flow in the following Fig. 1. It seemingly represents a type of embedding problem in iteration theory, when the embeddability of the given mapping $T_v(z)$ in real (metal) flows fails. We note that the attribute „icosahedral" is assigned not only by the relation (17) itself, but also by the known fact that no crystallographic group can be associated with the group A_5 [Li 2016].

Consequently, only the star $S_{\alpha}(k_j)$ related to the FE-mesh as formed physically by grain boundaries of colloidal crystal can represent some symmetry of the convergent model.

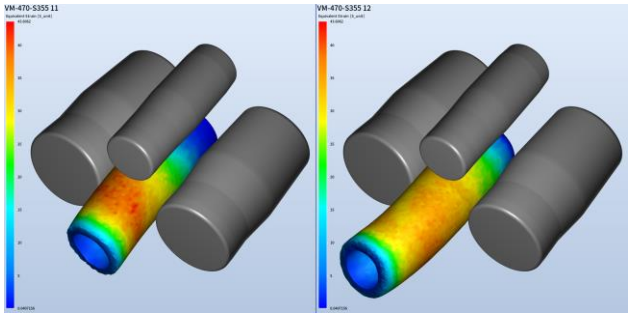

Figure 1. $S_{\alpha}(k_j)$ – symmetrical FEM-simulation of the MMP for two different values of the calibration constant *b*: for an incomplete phase separation (left) and for a complete phase separation (right) as assigned by solutions of the model (4).

If we consider "some symmetry" of the mathematical model, we talk about an existence of some symmetry of the polynomial $P(\chi)$ as well. Namely, as it immediately follows from (19), there is a set

$$\nu \notin S_M := \{2, 8/5, 336/223 \pm 4\sqrt{143}/223, 4/3, 20/21, 16/9\}. \tag{19a}$$

with no unifying value of the shear viscosity ν for *M*=7 which could be defined with respect to the group *G* as the union of seven proper subgroups (see also the Tomkinson´s theorem [Tomkinson 1997]). Thus, instead of *G* we have only a fiber (11) that unifies all seven instances of an occurrence of $S_{\alpha}(k_j)$ with respect to (12) with the vector k_j from no crystallographic system in 3D. In other words, the whole "symmetry" of polynomial (19) consists in a „quotient"

$$U/f_a \Leftrightarrow S_{\alpha}(k_j). \tag{20}$$

Then, this incomplete and consequently instable "symmetry" can easily land at the divergences (13) and (14) which are covered by the broken icosahedral symmetry [Roshal 2014] and its iterative scheme. As it follows from the divergence courses shown in Fig. **2** (calibration coupling constant correct, mesh choice incorrect) and **3** (coupling constant incorrect, mesh correct) below, such scheme is represented by the order 60 and subsequently 120 (imaginary axes of a divergence oscillations) of the full isosahedral group reflected via the Newton-Raphson iterations number [Ouypornpraset 2016].

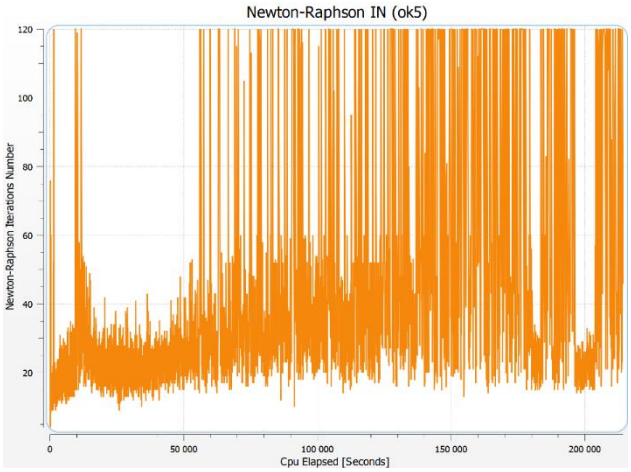

Figure 2. The divergence of icosahedral iterative scheme as emerged at MMP-FEM-simulation

Note. The peaks of the Newton-Raphson iterations number evolve from the order 60 of the group A_5 up to the order 120 of the group $A_5 \times Z_2$ during the CPU elapsed time. Let´s remind that the whole FEM-simulation (by means of the commercial software FORGE) has the mathematical model (4). This model can be attracted to have the full icosahedral group of symmetry in a dependance of how the choice of the FE-mesh is realized here. Namely, the MMP-calibration coupling constant *b* was involved correctly, contrary to the FE-mesh density choice. Consequently, the vector k_j was neither from any crystallographic system, nor from the corresponding colloidal crystal lattice.

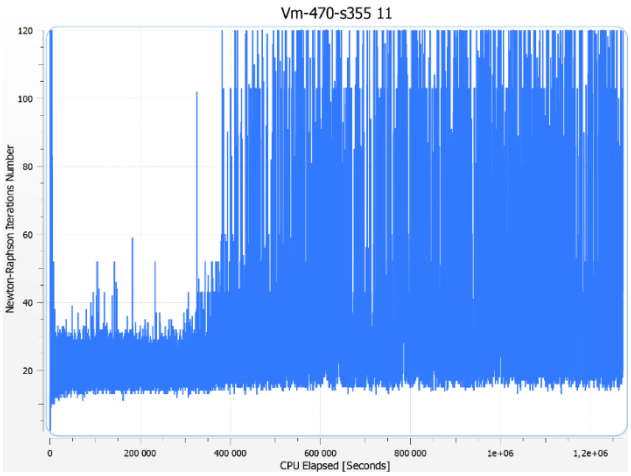

Figure 3. The divergence of icosahedral iterative scheme as emerged at MMP-FEM-simulation

Note. The peaks of the Newton-Raphson iterations number evolve only up to the order 120 (avoiding the order 60 to a large extent) of the group $A_5 \times Z_2$ during the CPU elapsed time. So, the nature of this „evolution" differs from the previous one. The

reason is an incorrect involving of the calibration coupling constant *b* within the model. Due to the fact that the numerical value of the parameter *a* was inadmissibly contained in a pair of some dyadic rationals generating additively a locally cyclic group which (contrary to Z_2) is avoided to stay in the direct product with A_5. Lattice symmetries resulting from the colloidal crystal are limited, but optimal at any vector k_j from a lattice coupled with the appropriate choice of the FE-mesh. If the density of the FE-mesh in the vicinity of the piercing plug is maximal, then also the stability of the „masked plug" problem (4) as the model of the FEM-simulation process becomes maximal. Such a case is namely most relevant with respect to the meaning of the Cahn-Hilliard equation, since the density of the FE-mesh in the vicinity of the piercing plug is separated from the rest of the FE-mesh density without separation of the „both" meshes. It is an unique case, when *the convergence of the $S_\alpha(k_j)$-appropriate iterative scheme* can exist, being most stable at some maximal difference between the both FE-mesh densities. Beyond, we can see this case in the Fig.4.

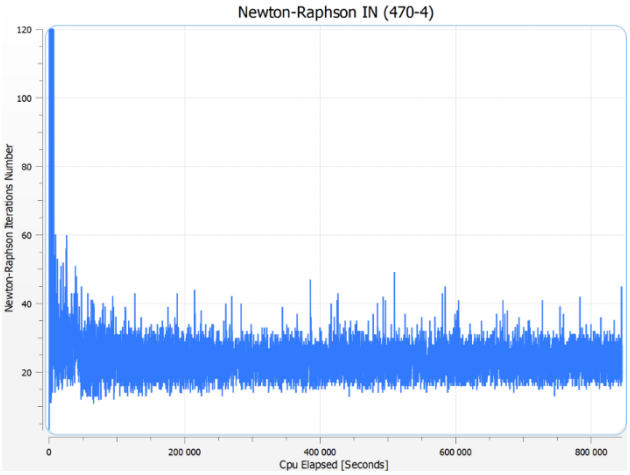

Figure 4. The convergence of the $S_\alpha(k_j)$ −appropriate iterative scheme as emerged at MMP-FEM-simulation

Note. The most remarkable feature of this convergence is that its peaks lie prevailingly at the value of

$$\mathrm{IN} \cong \tfrac{1}{2}|A_5| = 30\,, \tag{21}$$

where as $|A_5|$ is denoted the order of the group A_5.

According to the so-called Lagrange theorem [Gyamfi 2021], the order of any subgroup of some considered group divides the order of the group by an integer called as the index of the subgroup in the group. So, the above figure shows that there is no subgroup of A_5 and/or $A_5 \times Z_2$ respectively with the index 2, that is really the case. This directly demonstrates the correctness of our whole approach to the given FEM-simulation process.

In the following paragraph we will shortly study, how to obtain the singular solutions to the Navier-Stokes equations (NSE) yielding the star $S_\alpha(k_j)$ of the vector k_j , creating the Cahn-Hilliard-Navier-Stokes-system in the given context. (Another example of the Cahn-Hilliard-Navier-Stokes system as related to an appropriate FE-approximation scheme can be seen e.g. in [Kay 2008].)

3.1 The Cahn-Hilliard-Navier-Stokes system based on a pair of singular solutions to NSE

Considering the viscous flow of a colloidal solution between two rotating cylinders (Taylor-Couette system), we introduce a cylindrical surface $\partial\mathbf{U}$ (see U for (5)) between them separating mutually two phases in the colloidal crystal. - We expect that the Taylor vortex-flow gets, as the whole, into a certain state $\psi(b,s)$ on ∂U that subsequently bifurcates into binary components $\psi_1(k(x,t)\,s,\xi)$ and $\psi_2(k(x,t)\,s,\xi)$ at the point *x*, at which it is supposed that the vector k_j can be found in a time $t \in (0,T)$. – We talk about the vector of a norm *k* from a colloidal crystal lattice with respect to the so-called volume viscosity (the second viscosity) ξ corresponding with some singular value *q* of a coupling constant *b*. We assume that a fluid (colloidal solution) should be compressible in order to permit an emergence of a colloidal crystal. That´s why the Navier-Stokes equations should be taken in a form expressing the velocity vector w_i of the Taylor vortex-flow of *compressible* fluid (as given in e.g. [Hughes 1964]).

$$\rho[\partial_t w_i + w_j\partial_j w_i] = -\partial_i P + F_i + \partial_j[\nu(\partial_j w_i + \partial_i w_j - 2/3\delta_{ij}\partial_k w_k)] + \partial_i(\xi\partial_k w_k) \tag{22}$$

$$\left(\partial_j w_i + \partial_i w_j\right) = 0 \qquad \text{on} \qquad \partial\mathrm{U}. \tag{23}$$

The quantity *P* represents a required pressure, F_i a force density, δ_{ij} is the Kronecker delta and a factor 2/3 coincides with (11). – Respecting non-commutative relations

$$\psi_1 w_i \neq w_i\psi_1, \psi_2 w_j \neq w_j\psi_2 \text{ at } (\partial_j\psi_1 - \partial_i\psi_2) = 0 \text{ on } \partial\mathrm{U} \tag{24}$$

and using TP-Complex, we find that the both states cannot be any complex valued function, so that

$$\begin{aligned}\psi_1 = exp[&s(2k^4q^3 + k^3q^2(2q(s-1) - s + 3) \\ &+ k^2q(1-s)(q(s-3)+2) + k(1-s)(2q(s-1)+1) \\ &- (s-1)^2 2)/(\xi q^2(kq - s + 1))]\\ \psi_2 = exp[&k^3s(2q^3 + q^2(3-s) \\ &- 2q(s-1))/(\xi q^2(q - s + 1) + (k^2s(s-1)(2q^3 \\ &+ q^2(3-s) - 2q(s-1) - (s-1))/(\xi q^2(q - s + 1))]\end{aligned} \tag{25}$$

As we know, we have no crystallography or, more precisely, no Brillouin zone to be able to construct the star of the vector k_j within the lattice of the colloidal crystal directly. Therefore, we find only this star in a form of the state preventing the colloidal crystal from its possible deformations due to some interactions between the states ψ_1 and ψ_2. This form then reads

$$S_\alpha(k_j) = exp[-k^4/\xi + k^3/\xi - 5k^2/4\xi + k/2\xi - 1/4\xi] \tag{26}$$

It is already easy to show that (25) satisfies the relation (20) as

$$S_\alpha(k_j) \neq \tfrac{U}{f_a} \Leftrightarrow S_\alpha(k_j) \text{ for } \xi \in S_M,\ M = 7. \tag{27}$$

The relation (27) must hold in the above considered Cahn-Hilliard-Navier-Stokes system in order to preserve convergences like in the Fig. 4 at FEM-simulations of the MMP.

4 CONCLUSIONS

It seems to be for sure that any FEM-simulation of the MMP is physically realistic only then, if it has its own mathematical model connected with the original mathematical model of the atomic-level MMP. Connections are realized via the symmetries, namely the icosahedral one for the case of the divergence of the Newton Raphson number in the CPU-time and the quasi-symmetry (emerged as the star of the vector from the colloidal crystal lattice) parametrizing the FE-mesh choice for the convergence case. This star stays further as a picture of the configuration of perturbation mode coupled with cavity formations sequence during the MMP.

The mathematical model of the MMP-FEM-simulation (realized by the commercial software FORGE here) is a part of the complex Cahn-Hilliard-Navier-Stokes-system representing a complex algorithm coupled with the quintic determining the identity of the solved problem.

As a demonstration of the correctness of the given complex system some elementary theorems of the group theory were yielded naturally. Namely, the Tomkinson and Lagrange ones.The Tomkinson theorem states that if $\sigma(\boldsymbol{G})$ is defined as the smallest *N* such that G is the union of *N* proper subgroups, then there is no finite *G* satisfying $\sigma(G) = 7$. Since this theorem is directly implied by (8) for $Im(z) \neq 0$ in a contrast to the number $M = 7$ of occurrences of $S_{\alpha}(k_j)$. These seven disjoint stars are unified by means of the solution *U* of the equation (4), so this equation is directly compatible with the Tomkinson theorem by contradiction.

We have described the main blocks of our "know-how" of an approach to the FEM-simulations of processes, namely of the Mannesmann process-cavity formations here, as a part of the complex systems. Let´s note that data concerning technological parameters of the MMP setting principally, besides its calibration, do not play substantial role from the point of view of the structure and symmetries emerged within the whole algorithm.

The relationship between FEM-simulation and symmetry being meaningfully bounded by the group theory is surprising and quite novel. Applying the given algorithm at the considered FEM-simulation process, we save hundreds of computation hours.

ACKNOWLEDGMENTS

This work was supported by the EU Regional Development Fund within the Operational Programme Research, Development and Education under the aegis of Ministry of Education, Youth and Sports of the Czech Republic; Project number CZ.02.1.01/0.0/0.0/17_049/0008399.

REFERENCES

[Perna 2020] Perna, T. On One Controllability of the Schroedinger Equation as Coupled with the Atomic-Level Mannesmann Effect. Symmetry 2020, 12,1301.

[Pocta 1976] Pocta, B. Ocelove trubky, 1st ed.; SNTL: Praha, Czechoslovakia, 1976; p. 76. (In Czech)

[Berazategui 2006] Berazategui, D. A., Cavaliere, M. A., Montelatici, L., & Dvorkin, E. N. On the modelling of complex 3D bulk metal forming processes via the pseudo-concentration's technique. Application to the simulation of the Mannesmann piercing process. International journal for numerical methods in engineering, 2006, 65(7), 1113-1144.

[Pereira 2008] Pereira, A. L., & Pereira, M. C. An eigenvalue problem for the biharmonic operator on $\mathbb{Z}2$-symmetric regions. Journal of the London Mathematical Society, 2008, 77(2), 424-442.

[Tsankov 2011] Tsankov, T. The additive group of the rationales does not have an automatic presentation. The Journal of Symbolic Logic, 2011, 76(4), 1341-1351.

[Burger 2009] Burger, M., He, L., & Schoenlieb, C. B. Cahn–Hilliard inpainting and a generalization for gray value images. SIAM Journal on Imaging Sciences, 2009, 2(4), 1129-1167.

[Tateno 2019] Tateno, M., Tanaka,H. Numerical prediction of colloidal phase separation by direct computation of Navier-Stokes equations. Npj Computational Materials, 5, Article number: 40 (2019)

[Litzman 1982] Litzman, O., Sekanina, M. Uziti grup ve fyzice, ACADEMIA, Praha, Czechoslovakia, 1982 (In Czech)

[Larignon 2009] Larignon, B., Marr, K., & Goldstein, D. B. Monte Carlo and Navier-stokes simulations of compressible Taylor Couette flow. Journal of thermophysics and heat transfer, 2009, 20(3), 544-551

[Perna 1995] Perna, T. Algorithmic system for solving the fundamental PDEs. HMM TP-Complex. (The system is developed by Perna T.) 1995

[Meyer 2005] Meyer, Ch. The mirror quintic as a quintic. ArXiv:Math/0503329v1, [math.AG], 2005

[Wilson 2009] Wilson, R. The finite simple groups. Springer-Verlag, New York, 2009.

[Doyle 1989] Doyle, P., & McMullen, C. T. Solving the quintic by iteration. Acta Mathematica-Stockholm, 1989.

[Li 2016] Li, R., Li, A.Z., Dong, Z., Khor, K.A. A Review of Transmission Electron Microscopy of Quasicrystals – How Are Atoms Arranged? Crystals 6(9):105, 2016

[Tomkinson 1997] Tomkinson, M. J. Groups as the union of proper subgroups. Mathematica Scandinavica, 1997, 191-198.

[Roshal 2014] Roshal, D.S., Myasnikova, A.E., Rochal, S.B., Slightly broken icosahedral symmetry advances Thomson problem. arXiv:1408.3473 [cond-mat.soft], 2014

[Ouypornpraset 2016] Ouypornpraset, W. Beyond the Quadratic Equations and the N-D Newton-Raphson Method. IOSR Journal of Mathematics 12, August 2016, 44-76

[Gyamfi 2021] Gyamfi, B., Aidoo, A., Akweittey, E. Some Applications of Lagrange´s Theorem in Group Theory Using Numerical Examples.
WWJMRD 2021; 7(2): 32-34

[Kay 2008] Kay, D., Styles, V. Finite element approximation of a Cahn-Hilliard-Navier-Stokes system. Interfaces and Free Boundaries 10(1)
January 2008, 15-43.

[Hughes 1964] Hughes, W.F.; Gaylord, E.W. Basic equations of engineering science. Schaum´s outline series, McGraw-Hill Book Company.

CONTACTS:

Ing. Tomas Perna

MATERIAL AND METALLURGICAL RESEARCH, Ltd.

Pohranicni 693/31, Vitkovice 70300, Ostrava, Czech republic
projekty@mmvyzkum.cz, www.mmvyzkum.cz